# Change-of-Base Formula

## Change of Base Formula

Let $u$, $b$, and $c$ be positive numbers with $b \neq 1$ and $c \neq 1$. Then:

$$\log_c u = \frac{\log u}{\log c} \quad \text{and} \quad \log_c u = \frac{\ln u}{\ln c}$$

Evaluate each using common and natural logarithms.

a. $\log_2 8$

$\frac{\log 8}{\log 2} = \boxed{3}$

$\frac{\ln 8}{\ln 2} = 3$

b. $\log_4 8$

$\frac{\log 8}{\log 4} = \boxed{1.5}$

$\frac{\ln 8}{\ln 2} = 1.5$

c. $\log_3 25$

$\frac{\log 25}{\log 3} \approx 2.93$

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Change-of-Base WS 1

Name Key

Use a calculator to approximate each to the nearest thousandth.

1) $\log_3 3.3$

$\frac{\log 3.3}{\log 3} \approx 1.09$

2) $\log_2 30$

$\frac{\log 30}{\log 2} \approx 4.907$

3) $\log_4 5$

$\frac{\log 5}{\log 4} \approx 1.16$

4) $\log_2 2.1$

$\frac{\log 2.1}{\log 2} \approx 1.07$

5) $\log_{3.5} 5$

$\frac{\log 5}{\log 3.5} \approx 1.28$

6) $\log_6 13$

$\frac{\log 13}{\log 6} \approx 1.43$

7) $\log_6 40$

$\frac{\log 40}{\log 6} \approx 2.06$

8) $\log 43.5$

$\frac{\log 43.5}{\log 10} \approx 1.64$

9) $\log_2 2.9$

$\frac{\log 2.9}{\log 2} \approx 1.536$

10) $\log_6 22$

$\frac{\log 22}{\log 6} \approx 1.725$

11) $\log_7 8.7$

$\frac{\log 8.7}{\log 7} \approx$

12) $\log_3 62$

$\frac{\log 62}{\log 3} \approx 3.757$

Student Name: Key Score:

## Rewrite as Single Logarithm Using Change of Base Rule

*Example*: $\dfrac{\log_5 8}{\log_5 3} = \log_3 8$

$\dfrac{\log_2 15}{\log_2 6} =$ $\log_6 15$

$\dfrac{\log_4 9}{\log_4 5} =$ $\log_5 9$

$\dfrac{\log_3 8}{\log_3 7} =$ $\log_7 8$

$\dfrac{\log_{10} 5}{\log_{10} 2} =$ $\log_2 5$

$\dfrac{\log_{11} 6}{\log_{11} 2} =$ $\log_2 6$

$\dfrac{\log_3 10}{\log_3 5} =$ $\log_5 10$

# Doubling-Time & Half-Life

## 1 Exponential Growth & Decay

Recall the formula we developed for compound interest,

$$A = P\left(1 + \frac{r}{n}\right)^{nt},$$

where

$$\begin{aligned} A &= \text{accumulated amount of investment,} \\ P &= \text{initial investment,} \\ r &= \text{annual interest rate,} \\ n &= \text{number of compoundings per year,} \\ t &= \text{number of years.} \end{aligned}$$

We can rewrite the above formula in words as:

$$\begin{pmatrix} \text{Accumulated} \\ \text{Amount} \end{pmatrix} = \begin{pmatrix} \text{Initial} \\ \text{Amount} \end{pmatrix} \left[1 + \begin{pmatrix} \text{Rate of Growth} \\ \text{Per Period} \end{pmatrix}\right]^{\text{(Number of Periods)}}$$

The benefit of writing the formula in words is that we can now see our "monetary terms" from the compound interest formula have disappeared, so there is no reason the formula can't be applied to other types of problems involving amounts growing (or decaying – i.e., decreasing in amount). We now discuss Doubling-Time and Half-Life, which are two examples of how this formula can be applied to non-investment type problems.

## 2 Doubling-Time

Consider the following motivational problem:

> Problem 1 – Small rural water systems are often contaminated with bacteria by animals. Suppose that a water tank is infested with a colony of 100,000 *E. coli* bacteria. In this tank the colony doubles in number every 4 days. Determine a formula, $A(t)$, for the number of bacteria present in the tank after $t$ days.

We will proceed by using our formula,

$$\begin{pmatrix} \text{Accumulated} \\ \text{Amount} \end{pmatrix} = \begin{pmatrix} \text{Initial} \\ \text{Amount} \end{pmatrix} \left[1 + \begin{pmatrix} \text{Rate of Growth} \\ \text{Per Period} \end{pmatrix}\right]^{\text{(Number of Periods)}}.$$

The amount of time it takes the bacteria to double is 4 days, so this is our **period**. Since the bacteria doubles every 4 days, the amount of bacteria increases by 100% every 4 days (or each period). Hence, the **rate of growth per period** is 1.00 (corresponding to the 100% increase). The **number of periods** that have passed after $t$ days is given by $\frac{t}{4}$ (since our period is 4 days). Therefore, starting with an **initial amount** of 100,000 our formula becomes

$$\begin{aligned} A(t) &= (100,000)[1 + 1.00]^{\frac{t}{4}} \\ &= 100,000(2)^{\frac{t}{4}}. \end{aligned}$$

Log Word Problems WS

Name Key

Use the following formulas:

- Exponential Growth: $y = a(1+r)^t$
- Exponential Decay: $y = a(1-r)^t$
- Continuously Compounded Interest: $A = Pe^{rt}$

1. You buy a commemorative coin for $110. Each year t, the value V of the coin increases by 4%. How long will it take for the value of the coin to reach $200?

$\frac{200}{110} = \frac{110(1+.04)^t}{110}$ $\log_{1.04} 1.81 = \log_{1.04} 1.04^t$ $t = \log_{1.04} 1.81$ $\boxed{t \approx 15.1 \text{ yrs}}$

2. You deposit $1600 in a bank account. How long will it take for the value of the account to reach $2,000 if the account pays 4% annual interest compounded yearly.

$\frac{2000}{1600} = \frac{1600(1+.04)^t}{1600}$

$\log_{1.04} 1.25 = \log_{1.04} 1.04^t$ → $t = \log_{1.04} 1.25$ $\frac{\log 1.25}{\log 1.04}$

$\boxed{t \approx 5.69 \text{ yrs}}$

3. You purchased a plot of land that for $45,000. The value of the land increases by approximately 5% each year. How long will it take for the value of the land to double?

$90{,}000 = 45{,}000(1+.05)^t$ → $t = \log_{1.05} 2$

$\log_{1.05} 2 = \log_{1.05} 1.05^t$ $t = \frac{\log 2}{\log 1.05} \approx \boxed{14.2 \text{ yrs}}$

4. You buy a new car for $22,000. The value of the car decreases by 12.5% each year. When will the car have a value of $10,000?

$\frac{10{,}000}{22{,}000} = \frac{22{,}000(1-.125)^t}{22{,}000}$ $\log_{.875} .45 = \log_{.875} (.875)^t$ $t = \log_{.875} .45 \approx \boxed{5.97 \text{ yrs}}$

5. You buy a stereo system for $780. Each year the value of the stereo system decreases by 5%. When it was destroyed in a flood, your insurance only paid the depreciated value of $250. How long did you own the stereo before the flood?

$\frac{250}{780} = \frac{780(1-.05)^t}{780}$ $\log_{.95} .32 = \log_{.95} .95^t$ $t = \log_{.95} .32$ $t = \frac{\log .32}{\log .95} \approx \boxed{22.2 \text{ yrs}}$

6. You deposit $975 in an account that compounds interest continuously. After 5 years, the value of the account is $1132.79. What percent interest was the account paying?

$\frac{1132.79}{925} = \frac{925e^{r(5)}}{925}$ → $\frac{\ln 1.16}{5} = \frac{5r}{5}$

$\ln 1.16 = \ln e^{5r}$ $r = \frac{\ln 1.16}{5} \approx .029 \rightarrow \boxed{2.9\%}$

Answers: 1) 15.26 years 2) 5.7 years 3) 14.2 years 4) 5.9 years 5) 22 years 6) 3%

Assorted Log Word Problems WS 2 Name Key

1. The exponential growth model $P = 5344e^{0.012744t}$ approximates the world population (in millions) from 1990. According to this model, when will the world population reach 68 million?

2. The number N of bacteria in a culture is given by the model $N = 100e^{0.2197t}$ where t is the time in hours. Estimate the time required for the population to double in size?

$\ln 2 = \ln e^{.2197t}$ $\quad \frac{\ln 2}{.2197} = \frac{.2197t}{.2197}$ $\quad \boxed{t \approx 3.15 \text{ hrs}}$

3. A satellite has a radioisotope power supply. The power output in watts is given by the equation $P = 50e^{\frac{-t}{250}}$, where t is the time in days since the power supply was placed in service.
   a) How much power will be available at the end of one year?
   b) The equipment aboard the satellite requires 10 watts of power to operate properly. What is the operational life of the satellite?

a) $P = 50e^{-365/250} \approx 11.61 W$ $\quad$ b) $\ln \frac{1}{5} = \ln e^{-t/250}$ $\quad (-250)\ln \frac{1}{5} = -\frac{t}{250}(-250)$ $\quad \boxed{t \approx 402 \text{ days}}$

4. You buy a commemorative coin for $110. Each year t, the value V of the coin increases by 4%. In how many years will the coin be worth $250?

$\frac{250}{110} = \frac{110(1+.04)^t}{110}$ $\quad \log_{1.04} 2.3 = \log_{1.04} 1.04^t \rightarrow \log_{1.04} 2.3 = t$ $\quad \boxed{t \approx 21 \text{ yrs}}$

5. Suppose $500 is invested at 6% annual interest compounded continuously. When will the investment be worth $1000?

$1000 = 500e^{.06(t)}$ $\quad \ln 2 = \ln e^{.06t}$ $\quad \frac{\ln 2}{.06} = \frac{.06t}{.06}$ $\quad \boxed{t \approx 11.6 \text{ yrs}}$

6. An investment service promises to triple your money in 12 years. Assuming continuous compounding of interest, what rate of interest is needed?

$\ln 3 = \ln 1e^{r(12)}$ $\quad \frac{\ln 3}{12} = \frac{12(r)}{12}$ $\quad r \approx .0916 = \boxed{9.16\%}$

7. A piece of machinery valued at $250,000 depreciates at 12% per year by the fixed rate method. After how many years will the value have depreciated to $100,000?

$100{,}000 = 250{,}000(1-.12)^t$ $\quad t = \frac{\log .4}{\log .88} \approx \boxed{7.17 \text{ yrs}}$

$\log_{.88} .4 = \log_{.88}(.88)^t$

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Answers: 1) t = 18.9 → 2009 2) 3.15 hours 3a) 11.61 watts 3b) 402 days

4) 21 years 5) 11.6 years 6) 9.16% 7) 7.17 years